Newton-Raphson Method For Solving Nonlinear Equations
Part II - Solution To Black-Scholes Implied Volatility

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December, 2010

In Part I we derived the Newton-Raphson method for solving non-linear equations and used those methods to solve a simple break-even problem. In Part II we will use the methods developed in Part I to calculate the implied volatility of a call option using the Black-Scholes option pricing equation. We are given the following observed market parameters...

Table 1 - Observed Market Parameters:

S	=	Current stock price	=	\$10.00
K	=	Call option exercise price	=	\$12.00
T	=	Time to option expiration in years	=	2.00
d	=	Dividend yield	=	0.00
r	=	Risk-free rate	=	0.05

The current market price of the call option is \$1.9174. Our task is to calculate the option's volatility implied by the known call option price and the observed market parameters in Table 1.

Setting Up The Problem

We will use the Newton-Raphson method to calculate the volatility used to price the call option via the Black-Scholes equation. To do this we will need the first derivative of call price with respect to volatility. This derivative is known as Vega and is one the Black-Scholes equation's well known Greeks. The following table presents the legend of symbols that we will use throughout the remainder of this exercise...

Table 2 - Legend of Symbols:

σ	=	The actual volatility used to price the call option
$V(\sigma)$	=	The actual price of the call option as a function of the actual volatility
$\hat{\sigma}$	=	A guess as to the volatility used to price the call option
$V(\hat{\sigma})$	=	The calculated price of the call option using the volatility guess $(\hat{\sigma})$
$V'(\hat{\sigma})$	=	First derivative call price with respect to the volatility guess $(\hat{\sigma})$

The equation for option price as a function of the known volatility is...

$$V(\sigma) = 1.9174 \tag{1}$$

Whereas we know the call option price today we do not know the volatility used to calculate that price. The equation for option price using the Black-Scholes equation and our guess as to volatility is...

$$V(\hat{\sigma}) = SN[d1] - Ke^{-rt}N[d2] \tag{2}$$

Where...

$$d1 = \frac{\ln\frac{S}{K} + (r + \frac{1}{2}\hat{\sigma}^2)T}{\hat{\sigma}\sqrt{T}} \;\;;\;\; d2 = d1 - \hat{\sigma}\sqrt{T} \;\;;\;\; N[z] = \int\limits_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \delta z \;\; (*) \tag{3}$$

* Cumulative normal distribution function of the random variate z

As noted above we need Vega, which is the derivative of call price with respect to volatility. The derivative of call price Equation (2) with respect to the volatility guess is...

$$V'(\hat{\sigma}) = \frac{\delta V(\hat{\sigma})}{\delta \hat{\sigma}} = S\sqrt{T}e^{-dT}N'[d1] \tag{4}$$

Where...

$$N'[d1] = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d1^2} \tag{5}$$

We can now equate Equation (1), Equation (2) and Equation (4) via a Taylor Series Expansion of the first order where $\delta\,\hat{\sigma}$ is the difference between the actual volatility and the volatility guess. This equation is...

$$\begin{aligned} V(\sigma) &= V(\hat{\sigma}) + V'(\hat{\sigma})\,\delta\hat{\sigma} \\ &= V(\hat{\sigma}) + V'(\hat{\sigma})(\sigma - \hat{\sigma}) \\ &= V(\hat{\sigma}) + V'(\hat{\sigma})\sigma - V'(\hat{\sigma})\hat{\sigma} \end{aligned} \tag{6}$$

If we rearrange the equation above so as to solve for σ Equation (6) becomes...

$$\sigma = \frac{V(\sigma) - V(\hat{\sigma}) + V'(\hat{\sigma})\hat{\sigma}}{V'(\hat{\sigma})} \tag{7}$$

Solution via Newton-Raphson - We iterate equation (7) above until σ approximates $\hat{\sigma}$. Using an initial volatility guess of 10% implied volatility is determined in only three iterations as follows...

Iteration	$\hat{\sigma}$	$V(\sigma)$	$V(\hat{\sigma})$	$V'(\hat{\sigma})$	σ
1	0.1000	1.9174	0.2555	4.9504	0.4357
2	0.4357	1.9174	2.1165	5.5567	0.3999
3	0.3999	1.9174	1.9168	5.5890	0.4000

The implied volatility for the call option is **0.4000**.

Table 3 - Calculation Example Using Iteration 1:

$V(\sigma)$	=	1.9174 per Equation (1) above
d1	=	-0.5114
N[d1]	=	0.3045
N'[d1]	=	0.3500
d2	=	-0.6528
N[d2]	=	0.2569
Exp(-rT)	=	0.9048
Exp(-dT)	=	1.0000
Sqrt(T)	=	1.4142
$V(\hat{\sigma})$	=	(10.00)(0.3045)-(12.00)(0.9048)(0.2569) = 0.2555
$V'(\hat{\sigma})$	=	(10.00)(1.4142)(1.0000)(0.3500) = 4.9504
σ	=	((1.9174)-(0.2555)+(4.9504)(0.1000))/(4.9504) = 0.4357